TM-6011 - Modul 2: Model Matematik

Advanced Reservoir Simulation

Mathematical Background in Reservoir Simulation

Basic Flow Equation
Notation and Basic Operation

- Vector Notation
  - A Vector is a number, which includes several values
  - Most commonly in 3 dimensional space:

$$\text{vector } u = \vec{u} = \underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

- *Del* Notation, ∇ or gradient of a scalar.
  - A scalar could have a gradient which forms a vector.

$$\nabla p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix}$$

Algebraic Operation

- The common algebraic operation is inner product.
- Notation : $\underline{u} \cdot \underline{v} = (\underline{u},\underline{v})$
  - $(\underline{u},\underline{v}) = u_1v_1 + u_2v_2 + \ldots u_nv_n$
  - The product of inner product is a scalar.
- *Del* operator is to specify the divergence of a vector.

$$\text{divergence of } \underline{u} = \nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

  - The divergence of a vector is a scalar.

Chain Rules

- If $w = f(u_1, u_2, u_3, \ldots u_m)$ and $u = f(x_1, x_2, x_3, \ldots x_n)$

$$\frac{\partial w}{\partial x_i} = \frac{\partial w}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial w}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \ldots + \frac{\partial w}{\partial u_m}\frac{\partial u_m}{\partial x_i}$$

- If $\rho = f(p, T)$ and $p = f(x,y,z,t)$ then,

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \rho}{\partial T}\frac{\partial T}{\partial x}$$

- If $\rho = f(p)$ and $p = f(x,y,z,t)$, then

$$\frac{\partial \rho}{\partial x} = \frac{d\rho}{dp}\frac{\partial p}{\partial x}$$

Linearity

- $L(c_1u_1) = c_1L(u_1)$
- $L(u_1 + u_2) = L(u_1) + L(u_2)$
- $L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$

Diffusivity Equation

- Liquid: $$\nabla^2 p = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t}$$
  - Continuity Equation.. $$\nabla \bullet \rho\vec{u} = -\frac{\partial(\phi\rho)}{\partial t}$$
  - Darcy's Law: $$\vec{u} = -\frac{k}{\mu}\nabla p$$
  - Equation of State : $$\rho = f(p)$$
    - For isothermal condition(fluid) : $$c = -\frac{1}{V}\frac{dV}{dp} = \frac{1}{\rho}\frac{d\rho}{dp}$$
    - For rock: $$c_f = \frac{1}{\phi}\frac{d\phi}{dp}$$

The derivation .....

- Darcy's Law into Continuity equation:

$$\nabla \bullet \rho\left(-\frac{k}{\mu}\nabla p\right) = -\frac{\partial(\phi\rho)}{\partial t}$$

- Using Linearity operation: Constant (k/u)

$$\nabla \bullet \rho\nabla p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

- Using the del operator, the differentiation of

$$\rho\nabla \bullet \nabla p + (\nabla\rho)\bullet(\nabla p) = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t} \qquad \rho\nabla^2 p + (\nabla\rho)\bullet(\nabla p) = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

- The density gradient:

$$\nabla\rho = \frac{d\rho}{dp}\nabla p = c\rho\nabla p$$

Final Form

$$\rho\nabla^2 p + \cancel{c\rho(\nabla p)\bullet(\nabla p)} = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

$$\nabla^2 p = \frac{\mu}{k}\frac{1}{\rho}\frac{\partial(\phi\rho)}{\partial t}$$

$$\nabla^2 p = \frac{\mu}{k}\frac{1}{\rho}\left(\phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t}\right)$$

$$= \frac{\mu}{k}\frac{1}{\rho}\left(\phi\frac{d\rho}{dp}\frac{\partial p}{\partial t} + \rho\frac{d\phi}{dp}\frac{\partial p}{\partial t}\right)$$

$$= \frac{\mu}{k}\frac{1}{\rho}\left[\phi(c\rho)\frac{\partial p}{\partial t} + \rho(c_f\phi)\frac{\partial p}{\partial t}\right]$$

$$= \frac{\phi\mu}{k}[c + c_f]\frac{\partial p}{\partial t}$$

Laplacian of *p*

$$\nabla^2 p = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

PR

- Tugas Baca (review paper)
  - Diffusivity Equation for real liquids with gas in solution (SPE 23740)
  - A variable rate solution to the nonlinear diffusivity Gas Equation (SPE 145468)

Question:

- Is the final equation a linear P.D.E?
- If we assume as linear, the we can find analytical solutions.
  - 1D Linear: $$\frac{\partial^2 p}{\partial x^2} = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$
  - 2D Cartesian: $$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$
  - Radial geometry: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$ $$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

Real Gas Diffusivity Equation

- Continuity equation: $\nabla \bullet \rho\vec{u} = -\frac{\partial}{\partial t}(\phi\rho)$
- Darcy's Law: $\vec{u} = -\frac{k}{\mu}\nabla p$
- Gas density: $\rho = \frac{pM}{zRT}$

$$\nabla \bullet \frac{pM}{zRT}\left(-\frac{k}{\mu}\nabla p\right) = -\frac{\partial}{\partial t}\left(\phi\frac{pM}{zRT}\right) \longrightarrow \nabla \bullet k\frac{p}{z\mu}\nabla p = \frac{\partial}{\partial t}\left(\phi\frac{p}{z}\right)$$

- The equation is not linear: z, μ depend on *p*.

$$\mathrm{m}(p) = \int_{p_o}^{p}\frac{2p'}{z(p')\mu(p')}dp' \longrightarrow m(p) = 2\int_{p_o}^{p}\frac{p}{z\mu}dp$$

Mathematical Trick:

- The derivative of m with respect to any variable ξ

$$\frac{dm}{d\xi} = \frac{2p}{z\mu}\frac{dp}{d\xi} \quad and \quad \nabla m = \frac{2p}{z\mu}\nabla p$$

$$\nabla \bullet \frac{2p}{z\mu}\nabla p = \frac{2}{k}\frac{\partial}{\partial t}\left(\phi\frac{p}{z}\right) \longrightarrow \nabla \bullet \nabla m = \frac{2}{k}\frac{\partial}{\partial t}\left(\phi\frac{p}{z}\right)$$

$$\nabla^2 m = \frac{2}{k}\left[\phi\frac{d}{dp}\left(\frac{p}{z}\right) + \frac{p}{z}\frac{d\phi}{dp}\right]\frac{\partial p}{\partial t}$$

$$= \frac{2}{k}\frac{z\mu}{2p}\left[\phi\frac{d}{dp}\left(\frac{p}{z}\right) + \frac{p}{z}\frac{d\phi}{dp}\right]\frac{2p}{z\mu}\frac{\partial p}{\partial t}$$

$$= \frac{2}{k}\frac{\phi\mu}{2}\left[\frac{z}{p}\frac{d}{dp}\left(\frac{p}{z}\right) + \frac{1}{\phi}\frac{d\phi}{dp}\right]\frac{\partial m}{\partial t}$$

$$= \frac{2}{k}\frac{\phi\mu}{2}(c + c_f)\frac{\partial m}{\partial t} \longrightarrow \nabla^2 m = \frac{\phi\mu c_t}{k}\frac{\partial m}{\partial t}$$

Darcy Law
Henry Darcy in 1856

- Every flow equation has a Darcy eq..
- Observation: fluid flow rate is proportion to pressure gradient and not to fluid viscosity.

$$\nu = \frac{q}{A} = -\frac{k}{\mu}\frac{dp}{dx}$$

Classification of 2nd Order PDE

- General Form: $a\,u_{xx} + b\,u_{xy} + c\,u_{yy} = H(x, y, u, u_x, u_y)$
- Descriminant: $b^2 - 4ac$

Descriminant	Type	Example/Form
< 0	Elliptic	$u_{xx} + u_{yy} = 0$ Laplace's Equation $u_{xx} + u_{yy} = c$ (Poisson's Equation/ PSS)
= 0	Parabolic	$u_{xx} = u_t$ (Fourie's Equation/ Transient)
> 0	Hyperbolic	$u_{xx} - u_{yy} = 0$

More about Flow Equation

- General diffusivity equation:
  - For Liquid: $\nabla \bullet \left[\frac{k_o}{\mu_o B_o} \nabla p \right] = \frac{\partial}{\partial t} \left[\frac{\phi}{B_o} \right]$
  - For Gas : $\nabla \bullet \left[\frac{k_g}{\mu_g B_g} \nabla p \right] = \frac{\partial}{\partial t} \left[\frac{\phi}{B_g} \right]$

$$\nabla . p = \frac{\partial p}{\partial y} \nabla p$$

$$\nabla . \nabla p = \nabla^2 p = \frac{\partial^2 p}{\partial x} + \frac{\partial^2 p}{\partial y} + \frac{\partial^2 p}{\partial y}$$ (Linear Flow)

$$\nabla . \nabla p = \nabla^2 p = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}$$ (Radial Flow)

- New Normalization variable: $$p_p = \left[\frac{\mu B}{k}\right]_n \int_{p_{base}}^{p} \frac{k}{\mu B}\, dp$$

$$\nabla p_p = \frac{\partial p_p}{\partial p} \nabla p \longrightarrow \frac{\partial p_p}{\partial p} = \left[\frac{\mu B}{k}\right]_n \frac{k}{\mu B}$$

- Applying chain rule $$\frac{\partial p_p}{\partial t} = \frac{\partial p_p}{\partial p} \frac{\partial p}{\partial t}$$

$$\frac{\partial}{\partial t}\left[\frac{\phi}{B}\right] = \frac{\partial}{\partial p}\left[\frac{\phi}{B}\right] \frac{\partial p}{\partial t} \longrightarrow \frac{\partial}{\partial t}\left[\frac{\phi}{B}\right] = \frac{\phi}{B}\left[\frac{1}{\phi}\frac{\partial \phi}{\partial p} - \frac{1}{B}\frac{\partial B}{\partial p}\right]\frac{\partial p}{\partial t}$$

$$\frac{\partial}{\partial t}\left[\frac{\phi}{B}\right] = \frac{\phi c_t}{B}\frac{\partial p}{\partial t}$$

Homework:

- Proof:

$$\nabla^2 p_p = \frac{\phi \mu c_t}{k} \frac{\partial p_p}{\partial t}$$